A statistical analysis of the dual-mode CMA

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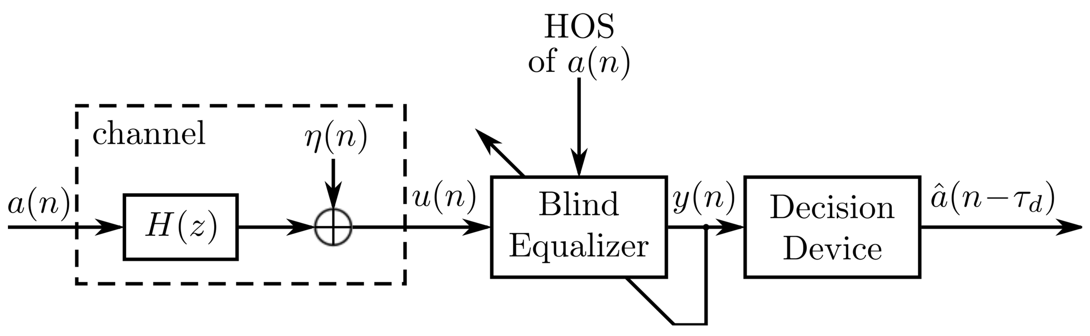
1. Introduction

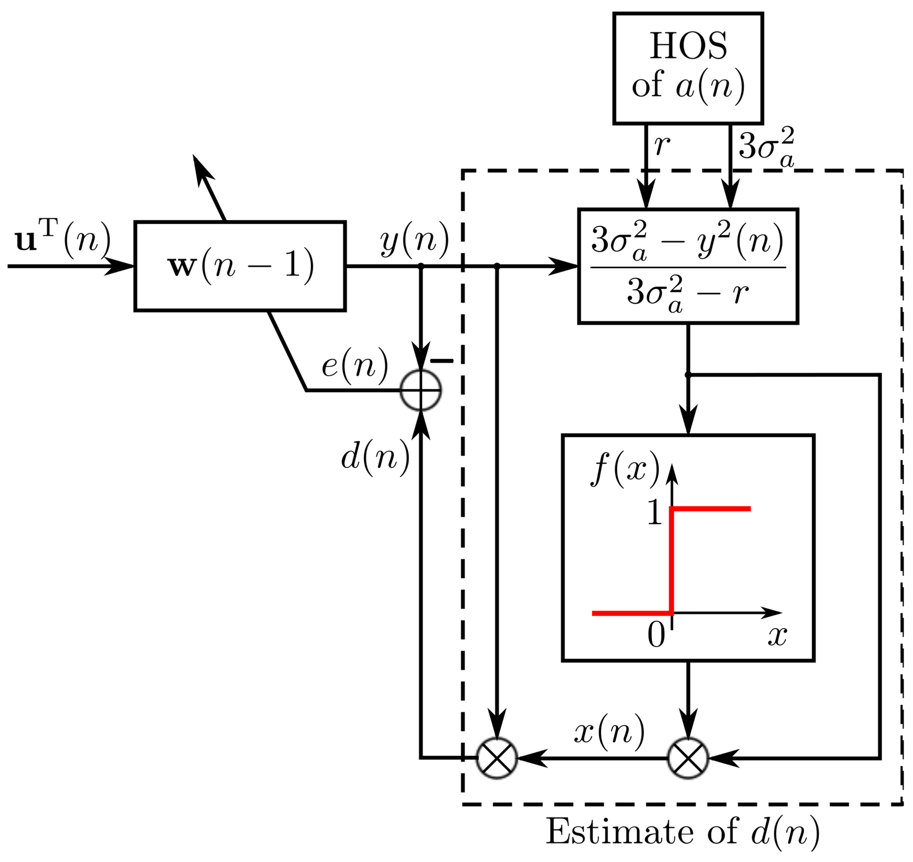
Blind equalization algorithms with good convergence and tracking properties and numerical robustness are desired to ensure the suitable performance of communications systems. In this paper, we present **transient** and **steady-state analyses** for the dual-mode constant modulus algorithm (**DM-CMA**), a version of CMA that **avoids its well-known divergence problem** [1]. We show that **DM-CMA is able to avoid divergence** without degradation of mean-square performance.

2. Problem formulation



Schematic representation of a communications system

Dual-mode CMA



Schematic representation of a DM-CMA equalizer

- Proposed in [1];
- $y(n) = \mathbf{u}^T(n)\mathbf{w}(n-1)$;
- $e(n) = \frac{[r-y^2(n)]y(n)}{\bar{\gamma}}$, $\quad r = \frac{\mathrm{E}\{a^4(n)\}}{\sigma_a^2}$, $\bar{\gamma} = 3\sigma_a^2 - r$;
- $e(n) = d(n) - y(n) \Rightarrow d(n) = x(n)y(n) = \frac{3\sigma_a^2 - y^2(n)}{3\sigma_a^2 - r}y(n)$;
- $y(n)$ and $d(n)$ are both estimates of $a(n-\tau_d)$;
- The consistency between $d(n)$ and $y(n)$ will be ensured if they have the same sign $\Rightarrow x(n) > 0$;
- The nonlinearity of the "error" signal of CMA is included in the factor $x(n)$;
- If $x(n) < 0 \Rightarrow$ outside region of interest (ROI)
  $d(n) \leftarrow 0$.
- $\mathbf{w}(n) = \mathbf{w}(n-1) + \frac{\mu}{\delta + ||\mathbf{u}(n)||^2}[d(n) - y(n)]\mathbf{u}(n)$.

3. Statistical models

Definitions

- unknown optimum coefficient vetor: $\mathbf{w}_o(n)$;
- weight-error vector: $\widetilde{\mathbf{w}}(n) = \mathbf{w}_o(n) - \mathbf{w}(n)$;
- *a priori* error $e_a(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}(n-1)$;
- EMSE: $\zeta(n) = \mathrm{E}\{e_a^2(n)\}$;
- autocorrelation matrix of the input: $\mathbf{R} = \mathrm{E}\{\mathbf{u}(n)\mathbf{u}^T(n)\}$;
- covariance matrix of the weight-error vector: $\mathbf{S} = \mathrm{E}\{\widetilde{\mathbf{w}}(n)\widetilde{\mathbf{w}}^T(n)\}$.

Main assumptions

A1 in a nonstationary environment, $\mathbf{w}_o$ follows a random walk model: $\mathbf{w}_o(n) = \mathbf{w}_o(n-1) + \mathbf{q}(n)$, $\mathbf{q}(n)$ i.i.d., $\mathbf{Q} = \mathrm{E}\{\mathbf{q}(n)\mathbf{q}^T(n)\}$;

A2 the optimal solution achieves perfect equalization, i.e.

$$a(n-\tau_d) \approx \mathbf{u}^T(n)\mathbf{w}_o(n-1) \Rightarrow y(n) \approx a(n-\tau_d) - e_a(n);$$

A3 $e_a^k(n)$, $k > 2$ are sufficiently small to be disregarded for $n \geq 0$, so that using **A2**, $e(n)$ can be approximated by

$$e(n) \approx \frac{\gamma(n)}{\bar{\gamma}}e_a(n) + \frac{\beta(n)}{\bar{\gamma}}$$

where $\gamma(n) = 3a^2(n-\tau_d) - r$ and $\beta(n) = ra(n-\tau_d) - a^3(n-\tau_d)$ are i.i.d. random variables;

A4 independence between the regressor vector $\mathbf{u}(n)$ and the weight-vector $\widetilde{\mathbf{w}}(n)$, which is widely used in the literature.

Transient analysis

- Using **A4** $\Rightarrow \zeta(n) \approx \mathrm{Tr}(\mathbf{RS}(n-1))$;
- Assuming that DM-CMA operates only inside ROI,

$$\widetilde{\mathbf{w}}(n) - \mathbf{q}(n) = \widetilde{\mathbf{w}}(n-1) - \frac{\mu}{\delta + \|\mathbf{u}(n)\|^2}e(n)\mathbf{u}(n);$$

- Using **A3** and assuming that the impulse response of the channel is long, we arrive at

$$\mathbf{S}(n) \approx \mathbf{S}(n-1) + \frac{\mu^2\xi\alpha_4}{\bar{\gamma}^2}\left[2\mathbf{RS}(n-1)\mathbf{R} + \mathbf{R}\mathrm{Tr}\left(\mathbf{RS}(n-1)\right)\right] - \mu\alpha_2\left[\mathbf{S}(n-1)\mathbf{R} + \mathbf{RS}(n-1)\right] + \frac{\mu^2\sigma_\beta^2\alpha_4}{\bar{\gamma}^2}\mathbf{R} + \mathbf{Q}, \qquad (\bigstar)$$

where $\alpha_2 \triangleq \left[\sigma_u^2(M-2)\right]^{-1}$, $\alpha_4 \triangleq \left[\sigma_u^4(M-2)(M-4)\right]^{-1}$, ξ and σ_β^2 are constants that depend on HOS of $a(n)$.

Steady-state analysis

Inside ROI

- Traditional method:
  - $\zeta(\infty)$ can be obtained by calculating the trace of both sides of ($\bigstar$) when $n \to \infty$;
  - to arrive at an easy-to-compute expression, we assume that $2\mathrm{Tr}(\mathbf{RS}(\infty)\mathbf{R})$ can be disregarded in relation to $\mathrm{Tr}(\mathbf{R})\mathrm{Tr}(\mathbf{RS}(\infty))$.

$$\zeta(\infty) \approx \frac{\mu\bar{\gamma}^{-1}\sigma_\beta^2\alpha_4\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\bar{\gamma}\mathrm{Tr}(\mathbf{Q})}{2\bar{\gamma}\alpha_2 - \mu\bar{\gamma}^{-1}\xi\alpha_4\mathrm{Tr}(\mathbf{R})}. \qquad (\blacktriangle)$$

- Energy conservation:
  - Using the energy-conservation arguments, the EMSE can be obtained by calculating a recursion for $\widetilde{\mathbf{w}}(n)$.

$$\zeta(\infty) = \frac{\mathrm{Tr}(\mathbf{R})\left[\mu\bar{\gamma}^{-1}\sigma_\beta^2\alpha_2 + \mu^{-1}\bar{\gamma}\mathrm{Tr}(\mathbf{Q})\right]}{2\bar{\gamma} - \mu\bar{\gamma}^{-1}\xi} \qquad (\blacktriangle)$$

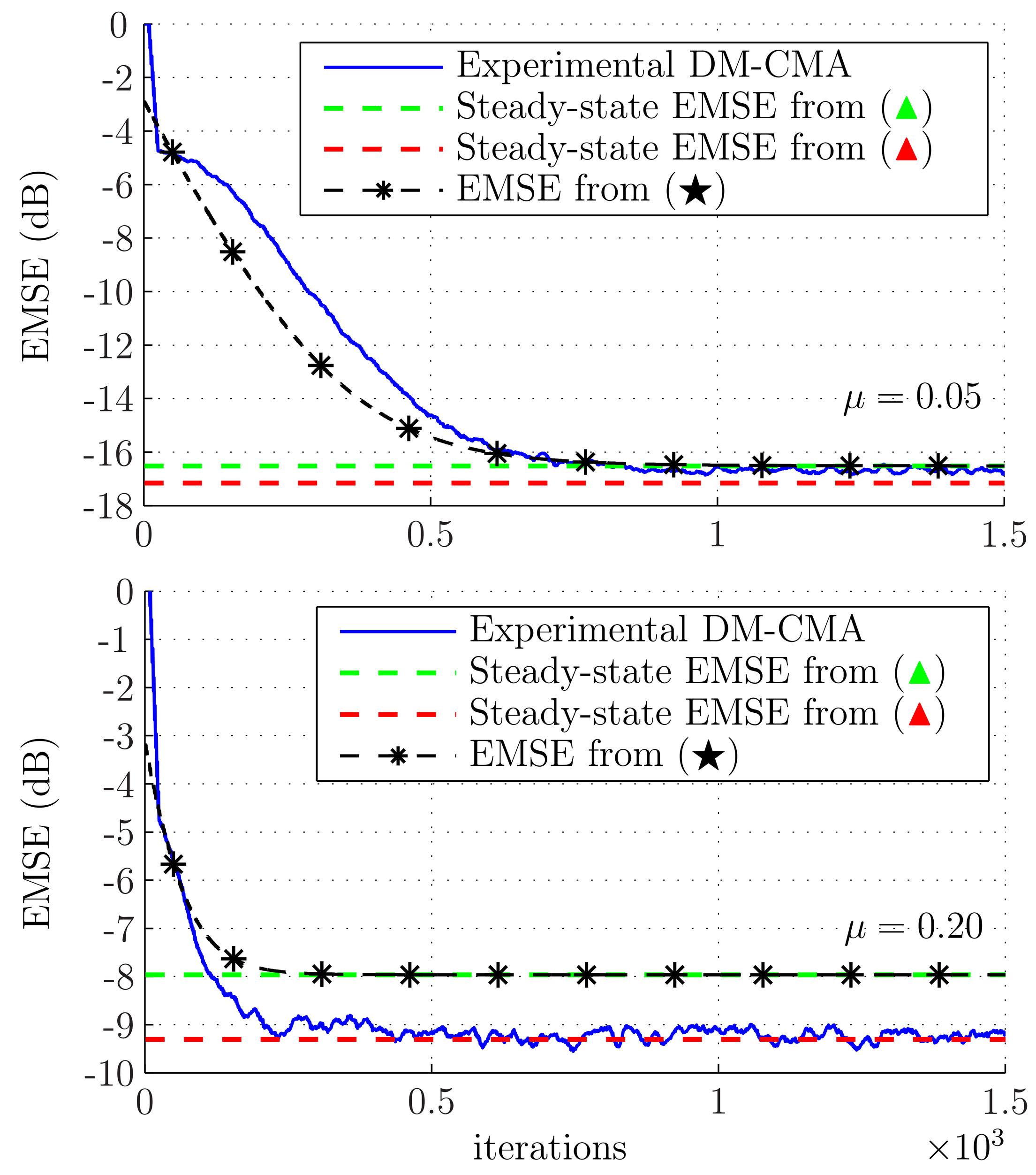
Outside ROI

- $d(n) = 0 \Rightarrow e(n) = -y(n)$;
- Since this mode of operation makes the algorithm return to the ROI, the result obtained here is a worst case analysis. Considering the energy-conservation arguments, we arrive at

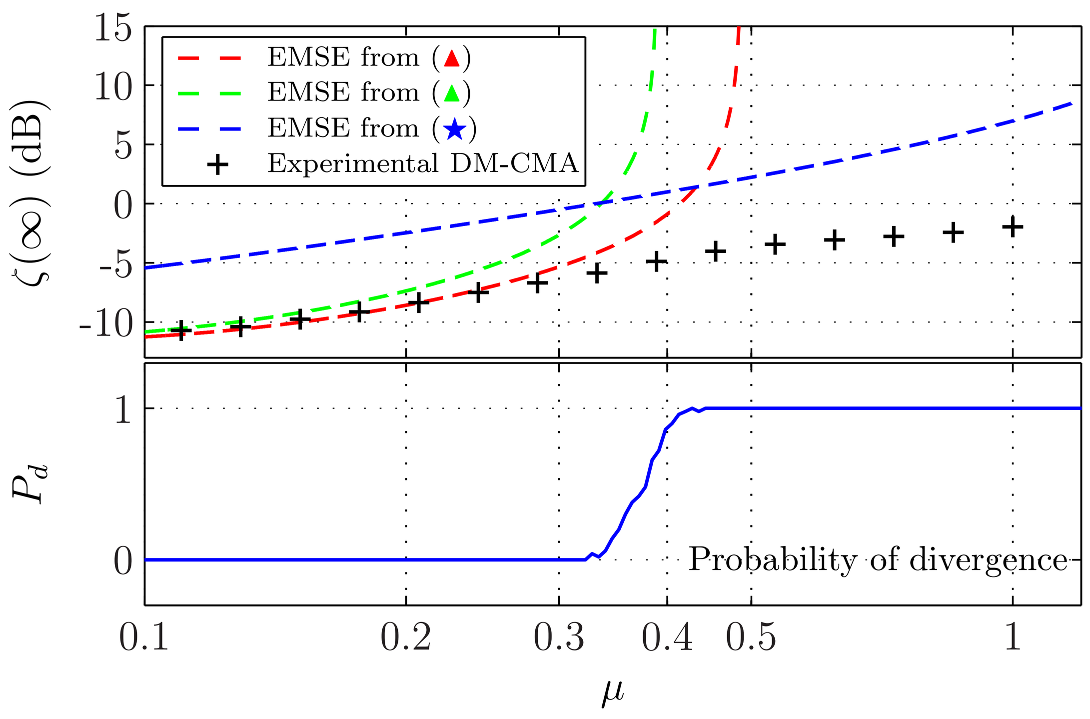
$$\zeta(\infty) = \frac{\mathrm{Tr}(\mathbf{R})[\mu\sigma_a^2\alpha_2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})]}{2-\mu}. \qquad (\bigstar)$$

4. Simulations

- Transmission of a 4-PAM signal;
- Channel $\mathbf{h} = [0.25\ 0.64\ 0.80\ -0.55]^T$;
- $M = 20$ coefficients, $T/2$-FSE.



Theoretical and experimental EMSE along the iterations for DM-CMA; $\mathbf{Q} = \mathbf{0}$; 500 independent runs.



Theoretical and experimental steady-state EMSE for DM-CMA; $\mathbf{Q} = 10^{-6}\mathbf{R}$; 50 independent runs.

5. Conclusions

- ($\bigstar$) shows a **good agreement** with simulations, mainly for **small step-sizes**;
- ($\blacktriangle$) provides a reasonable estimate for the **range of step-sizes** in which the probability of divergence of NCMA is approximately zero;
- ($\blacktriangle$) is more accurate for **larger step-sizes**;
- ($\bigstar$) in conjunction with ($\blacktriangle$) or ($\blacktriangle$) give a **range of values for the steady-state EMSE of DM-CMA** in all possible situations.

[1] M. D. Miranda, M. T. M. Silva and V. H. Nascimento, "Avoiding divergence in the constant modulus algorithm". Proc. of ICASSP 2008.

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