A statistical analysis of the dual-mode CMA

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***Abstract*— Blind equalization algorithms with good convergence and tracking properties and numerical robustness are desired to ensure the good performance of communications systems. In this paper, we present transient and steady-state analyses for the dual-mode constant modulus algorithm (DM-CMA), a version of CMA that avoids its well-known divergence problem. We show that DM-CMA is able to avoid divergence without degradation of mean-square performance. Good agreement between analytical and simulation results is observed.**

I. Introduction

Modern digital communications systems employ blind equalizers in order to remove the intersymbol interference introduced by dispersive channels. These equalizers avoid the repeated transmission of training signals, optimizing the use of the channel capacity [1]. A simplified communications system with a blind equalizer is depicted in Fig. 1. The signal $a(n)$, assumed independent, identically distributed, and non Gaussian, is transmitted through an unknown channel, whose model is constituted by a finite impulse response filter $H(z)$ and additive white Gaussian noise $\eta(n)$. From the received signal $u(n)$ and the known statistical properties of the transmitted signal, the blind equalizer must mitigate the channel effects and recover the signal $a(n)$ for some delay τ_d. The output of the equalizer is given by $y(n) = \mathbf{u}^T(n)\mathbf{w}(n-1)$, where $\mathbf{u}(n)$ is the input regressor vector, $\mathbf{w}(n-1)$ is the equalizer weight vector (both column vectors with M coefficients), and the superscript T denotes the transpose of a vector. It is also usual to assume that the channel is time-invariant and the sequences $\{a(n)\}$, $\{u(n)\}$, and $\{\eta(n)\}$ are stationary and have zero mean.

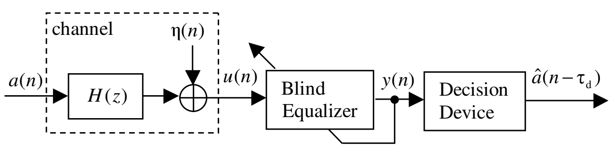

Fig. 1. Schematic representation of a communications system.

The constant modulus algorithm (CMA) [2], [3] is the most used algorithm for the adaptation of the blind equalizers. Its largest advantage is its small computational cost, however, it presents convergence problems since an inadequate choice of the step-size in conjunction with an initialization distant from the zero-forcing solution can lead it to diverge (i.e., the norm of the weight vector goes to infinity) or to converge to undesirable local minima [1]. Therefore, the convergence and stability of constant-modulus-based algorithms have been the subject of research for many years (see, e.g., [1], [4]–[6] and their references). In this context, analytical expressions for the excess mean square error (EMSE) of these algorithms have been computed in the literature (see, e.g., [4], [7]–[11]). Recently, a model for the estimation error of constant-modulus-based algorithms was proposed in [10] and extended in [6] to obtain stability conditions for CMA. For a range of step-sizes, [6] observed that CMA may diverge or not in a given run, with a probability of divergence that depends on how close the initial condition is to a local minimum, the step-size, and the noise level. Although these results are important to understand the CMA behavior, they do not solve the divergence problem in practical situations, since the local minima of the constant-modulus cost-function are unknown.

In order to avoid divergence, [12] proposed a modified version of CMA with two distinct operation modes. In the first mode, the algorithm works as a normalized CMA (NCMA), i.e., the coefficients are adapted as

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \frac{\mu}{\delta + \|\mathbf{u}(n)\|^2}[d(n) - y(n)]\mathbf{u}(n), \quad (1)$$

where $0 < \mu < 2$ is a step-size, δ is a regularization factor, $\|\cdot\|$ represents the Euclidean norm,

$$d(n) = x(n)y(n) = \frac{3\sigma_a^2 - y^2(n)}{3\sigma_a^2 - r}y(n), \quad (2)$$

$r = \mathrm{E}\{a^4(n)\}/\mathrm{E}\{a^2(n)\}$, $\sigma_a^2 = \mathrm{E}\{a^2(n)\}$, and $\mathrm{E}\{\cdot\}$ denotes the expectation operation. The update equation was written conveniently as (1) in order to include the nonlinearity of the "error" signal of CMA in the factor $x(n)$. Thus, (1) has the same structure as the normalized least mean-square (NLMS) algorithm. The difference is that $d(n)$ and $y(n)$ are both estimates of the transmitted signal. When the algorithm operates in this mode, it works in what is called *region of interest* (ROI) and can reach a stationary point of the cost function.

The consistency between the estimates $d(n)$ and $y(n)$ will be ensured if they have the same sign, which is equivalent to requiring the correction factor $x(n)$ to be always positive. Since the denominator of $x(n)$ is positive for practical constellations used in communications systems [12], $x(n) \geq 0$ occurs when $y^2(n) \leq 3\sigma_a^2$. On the other hand, if $y^2(n) > 3\sigma_a^2$, the algorithm leaves the ROI and enters the second operation mode. In this mode, the estimate $d(n)$ is simply rejected, i.e., we force $d(n) = 0$ and (1) reduces to

$$\mathbf{w}(n) = \mathbf{w}(n-1) - \frac{\mu}{\delta + \|\mathbf{u}(n)\|^2}y(n)\mathbf{u}(n). \quad (3)$$

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The algorithm with these two operation modes is called dual-mode CMA (DM-CMA). In [12], it was shown for scalar filters that (3) makes the algorithm return to the ROI. In the vector case, the good performance of the algorithm was confirmed through numerical simulations. Recently, the same idea was used in [13] to avoid divergence in the Shalvi-Weinstein algorithm.

In this work, we present a statistical analysis for DM-CMA. We first revisit the model for the estimation error of CMA of [10]. Then a transient analysis assuming that DM-CMA is inside the ROI is presented. We also present three steady-state analyses: two of them considering the operation inside the ROI and one worst case analysis assuming that DM-CMA operates only outside the ROI. The proposed model predicts situations in which the probability of divergence of NCMA is high. In order to simplify the arguments, we assume that all the quantities are real. We also employ $T/2$-fractionally-spaced equalizers (FSE), which ensure perfect equalization in a noise-free environment, under certain well-known conditions [14].

II. A statistical model for the estimation error

One measure of the equalizer performance is given by the excess mean-square error, defined as $\zeta(n) \triangleq \mathrm{E}\{e_a^2(n)\}$, where $e_a(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}(n-1)$ is the *a priori* error, $\widetilde{\mathbf{w}}(n) = \mathbf{w}_\mathrm{o}(n) - \mathbf{w}(n)$ is the weight-error vector, and $\mathbf{w}_\mathrm{o}$ is the optimal *zero-forcing* solution. We assume that in a non-stationary environment, the variation in $\mathbf{w}_\mathrm{o}$ follows a random-walk model (see, e.g., [15, p. 359]), that is, $\mathbf{w}_\mathrm{o}(n) = \mathbf{w}_\mathrm{o}(n-1)+\mathbf{q}(n)$. In this model, $\mathbf{q}(n)$ is an i.i.d. vector with positive-definite autocorrelation matrix $\mathbf{Q} = \mathrm{E}\{\mathbf{q}(n)\mathbf{q}^T(n)\}$ and is assumed independent of the initial conditions $\{\mathbf{w}_\mathrm{o}(-1), \mathbf{w}(-1)\}$ and of $\{\mathbf{u}(l)\}$ for all l.

A statistical analysis of constant-modulus-based algorithms requires simplifying assumptions. Among them, it is common to assume that

A1. the constellation used to generate $a(n)$ has circular symmetry, so that $\mathrm{E}\{a^k(n)\} = 0$ for all odd integers $k > 0$, which is true for practical constellations;

A2. the channel noise power is small enough for the zero-forcing solution $\mathbf{w}_\mathrm{o}$ to be one of the global minimizers of the constant-modulus cost function. In other words, the optimal solution achieves perfect equalization, i.e., $a(n-\tau_d) \approx \mathbf{u}^T(n)\mathbf{w}_\mathrm{o}(n-1)$ [4], [6], [8], [10]. Hence, the filter output can be approximated by

$$y(n) \approx a(n-\tau_d) - e_a(n). \tag{4}$$

Defining the estimation error as $e(n) \triangleq d(n) - y(n)$, replacing $y(n)$ by (4), and assuming that terms depending on $e_a^k(n)$, $k \geq 2$ are sufficiently small to be disregarded for all $n \geq 0$, after some algebra, we obtain

$$e(n) \approx \frac{\gamma(n)}{\bar{\gamma}} e_a(n) + \frac{\beta(n)}{\bar{\gamma}}, \tag{5}$$

where $\gamma(n) = 3a^2(n-\tau_d) - r$ and $\beta(n) = r\,a(n-\tau_d) - a^3(n-\tau_d)$. We should notice that $\gamma(n)$ and $\beta(n)$ are i.i.d. random variables, satisfying $\bar{\gamma} \triangleq \mathrm{E}\{\gamma(n)\} = 3\sigma_a^2 - r$, $\mathrm{E}\{\beta(n)\} = 0$,

$$\xi \triangleq \mathrm{E}\{\gamma^2(n)\} = 3\,r\,\sigma_a^2 + r^2, \quad \text{and} \tag{6}$$

$$\sigma_\beta^2 \triangleq \mathrm{E}\{\beta^2(n)\} = \mathrm{E}\{a^6(n) - r^2\,a^2(n)\}. \tag{7}$$

Assuming $\gamma(n) \equiv 1$ and interpreting $\beta(n)$ as a measurement noise, (5) reduces to the regression linear model used in supervised adaptive filtering [15, p. 284]. It is relevant to notice that $\beta(n)$ is identically zero for constant-modulus constellations, so the variability in the modulus of $a(n)$ (as measured by $\beta(n)$) plays the role of measurement noise for constant-modulus based algorithms [6], [10]. In addition, $\beta(n)$ is uncorrelated with $\mathbf{u}(n)$ as shown in [6, Lemma 1].

III. Transient analysis

It is common in the literature to approximate the EMSE as $\zeta(n) \approx \mathrm{Tr}(\mathbf{RS}(n-1))$, where $\mathbf{S}(n) \triangleq \mathrm{E}\{\widetilde{\mathbf{w}}(n)\widetilde{\mathbf{w}}^T(n)\}$ is the covariance matrix of the weight-error vector, $\mathbf{R} \triangleq \mathrm{E}\{\mathbf{u}(n)\mathbf{u}^T(n)\}$ is the autocorrelation matrix of the input signal, and $\mathrm{Tr}(\mathbf{A})$ stands for the trace of matrix $\mathbf{A}$. This approach is based on the independence assumption between the regressor vector $\mathbf{u}(n)$ and weight-error vector $\widetilde{\mathbf{w}}(n-1)$, and is justified for small step-sizes due to the different time-scales for variations in $\mathbf{u}(n)$ and $\widetilde{\mathbf{w}}(n-1)$ [16].

Assuming that DM-CMA operates inside the ROI, the recurrent equation for the weight-error vector can be obtained by subtracting both sides of (1) from $\mathbf{w}_\mathrm{o}(n)$ and recalling that $\mathbf{w}_\mathrm{o}(n) = \mathbf{w}_\mathrm{o}(n-1) + \mathbf{q}(n)$, i.e.,

$$\widetilde{\mathbf{w}}(n) - \mathbf{q}(n) = \widetilde{\mathbf{w}}(n-1) - \frac{\mu}{\delta + \|\mathbf{u}(n)\|^2} e(n)\mathbf{u}(n). \tag{8}$$

Replacing $e(n)$ by (5) and assuming that δ is small compared to $\|\mathbf{u}(n)\|^2$, (8) can be rewritten as

$$\widetilde{\mathbf{w}}(n) \approx \overbrace{\left[\mathbf{I} - \frac{\mu\gamma(n)}{\bar{\gamma}\|\mathbf{u}(n)\|^2}\mathbf{u}(n)\mathbf{u}^T(n)\right]\widetilde{\mathbf{w}}(n-1)}^{\mathbf{p}_1} - \underbrace{\frac{\mu\beta(n)}{\bar{\gamma}\|\mathbf{u}(n)\|^2}\mathbf{u}(n)}_{\mathbf{p}_2} + \underbrace{\mathbf{q}(n)}_{\mathbf{p}_3}. \tag{9}$$

To proceed, we also assume that

A3. $\beta(n)$ and $\gamma(n)$ are independent of $\widetilde{\mathbf{w}}(n-1)$ [4], [10];

A4. the impulse response of the channel is long enough so that [6], [10], [16], [17]:

$$\mathrm{E}\{\gamma(n)u^2(n)\} \approx \bar{\gamma}\,\mathrm{E}\{u^2(n)\}, \tag{10}$$

$$\mathrm{E}\{\beta^2(n)u^2(n)\} \approx \sigma_\beta^2\mathrm{E}\{u^2(n)\}, \tag{11}$$

$$\mathrm{E}\left\{\mathbf{u}(n)\mathbf{u}^T(n)\mathbf{S}(n-1)\mathbf{u}(n)\mathbf{u}^T(n)\right\} \approx 2\mathbf{RS}(n-1)\mathbf{R} + \mathbf{R}\mathrm{Tr}(\mathbf{RS}(n-1)), \tag{12}$$

$$\mathrm{E}\left\{\mathbf{u}(n)\mathbf{u}^T(n)/\|\mathbf{u}(n)\|^2\right\} \approx \alpha_2\mathbf{R}, \quad \text{and} \tag{13}$$

$$\mathrm{E}\left\{\mathbf{u}(n)\mathbf{u}^T(n)/\|\mathbf{u}(n)\|^4\right\} \approx \alpha_4\mathbf{R}, \tag{14}$$

where $\alpha_2 \triangleq \left[\sigma_u^2(M-2)\right]^{-1}$, $\alpha_4 \triangleq \left[\sigma_u^4(M-2)(M-4)\right]^{-1}$, and σ_u^2 is the variance of the input signal. The approximations (10) and (11) were used in the CMA analyses of [10] and [6]. Assuming that the regressor $\mathbf{u}(n)$ is Gaussian, (12), (13), and

(14) hold. The approximations (13) and (14) were obtained in the NLMS analysis of [17] and are valid for a large number of coefficients (e.g., $M \geq 20$).

Multiplying both sides of (9) from the right by their respective transposes and taking the expectation we obtain a recurrent expression for $\mathbf{S}(n)$. Using A1-A4 and the fact that $\mathbf{q}(n)$ is independent of the initial conditions and of $\mathbf{u}(n)$, we can observe that $\mathrm{E}\{\mathbf{p}_i\mathbf{p}_j^T\} = 0$ for $i, j = 1, 2, 3$ and $i \neq j$, where the $\mathbf{p}_i$ were defined in (9). Hence, we arrive at

$$\boxed{\begin{aligned}\mathbf{S}(n) \approx \mathbf{S}(n-1) + \frac{\mu^2\xi\alpha_4}{\bar{\gamma}^2}\left[2\mathbf{R}\mathbf{S}(n-1)\mathbf{R} + \mathbf{R}\mathrm{Tr}\left(\mathbf{R}\mathbf{S}(n-1)\right)\right] \\ -\mu\alpha_2\left[\mathbf{S}(n-1)\mathbf{R} + \mathbf{R}\mathbf{S}(n-1)\right] + \frac{\mu^2\sigma_\beta^2\alpha_4}{\bar{\gamma}^2}\mathbf{R} + \mathbf{Q}. \qquad (15)\end{aligned}}$$

IV. Steady-state analyses

In the following, we present three tracking analyses for DM-CMA. In the first two analyses, we assume that the algorithm operates inside the ROI and in the third one, we assume that it operates outside the ROI.

A. Traditional method - inside the ROI

An analytical expression for the steady-state EMSE can be obtained by calculating the trace of both sides of (15) when $n \to \infty$. To arrive at an easy-to-compute expression, we assume that the term $2\mathrm{Tr}(\mathbf{R}\mathbf{S}(\infty)\mathbf{R})$ can be disregarded in relation to $\mathrm{Tr}(\mathbf{R})\mathrm{Tr}(\mathbf{R}\mathbf{S}(\infty))$ [16]. Thus, after some algebra, we get

$$\boxed{\zeta(\infty) \approx \frac{\mu\bar{\gamma}^{-1}\sigma_\beta^2\alpha_4\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\bar{\gamma}\mathrm{Tr}(\mathbf{Q})}{2\bar{\gamma}\alpha_2 - \mu\bar{\gamma}^{-1}\xi\alpha_4\mathrm{Tr}(\mathbf{R})}. \qquad (16)}$$

B. Energy conservation - inside the ROI

Using the energy conservation arguments of [15, Ch. 7], an analytical expression for the steady-state EMSE inside the ROI can be obtained by equating the squared norms on both sides of (8) and taking the expectation as $n \to \infty$. Since (8) has the same structure as the NLMS algorithm, DM-CMA satisfies the same variance relation of [15, Th. 7.4.1]. Thus, for $\delta \approx 0$, we obtain

$$\mu\mathrm{E}\left\{\frac{e^2(n)}{\|\mathbf{u}(n)\|^2}\right\} + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left\{\frac{e_a(n)e(n)}{\|\mathbf{u}(n)\|^2}\right\}. \qquad (17)$$

Assuming that at steady-state

$$\mathrm{E}\left\{\frac{e_a^2(n)}{\|\mathbf{u}(n)\|^2}\right\} \approx \frac{\zeta(\infty)}{\mathrm{Tr}(\mathbf{R})} \qquad (18)$$

and using Model (5), the EMSE can be approximated by

$$\boxed{\zeta(\infty) = \frac{\mathrm{Tr}(\mathbf{R})\left[\mu\bar{\gamma}^{-1}\sigma_\beta^2\alpha_2 + \mu^{-1}\bar{\gamma}\mathrm{Tr}(\mathbf{Q})\right]}{2\bar{\gamma} - \mu\bar{\gamma}^{-1}\xi} \qquad (19)}$$

with $\mathrm{E}\left\{1/\|\mathbf{u}(n)\|^2\right\} \approx \alpha_2 \triangleq \left[\sigma_u^2(M-2)\right]^{-1}$. For $\gamma(n) \equiv 1$, (19) reduces to the analytical expression of the tracking EMSE of NLMS [15, Eq. (7.6.7)].

C. Energy conservation - outside the ROI

We now use (3) to analyze the case in which DM-CMA operates outside the ROI. Since (3) makes the algorithm return to the ROI, DM-CMA operates in this mode only for a finite-time interval. Therefore, the analytical expression obtained here is the result of a worst case analysis. Using energy conservation arguments, the steady-state variance relation outside the ROI can be obtained replacing $e(n)$ by $-y(n)$ in (17), i.e.,

$$\mu\mathrm{E}\left\{\frac{y^2(n)}{\|\mathbf{u}(n)\|^2}\right\} + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = -2\mathrm{E}\left\{\frac{e_a(n)y(n)}{\|\mathbf{u}(n)\|^2}\right\}. \qquad (20)$$

Using the approximations (4) and (18) in (20), the steady-state EMSE outside the ROI can be approximated by

$$\boxed{\zeta(\infty) = \frac{\mathrm{Tr}(\mathbf{R})[\mu\sigma_a^2\alpha_2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})]}{2-\mu}. \qquad (21)}$$

V. Simulation results

To verify the validity of the analyses, we consider in all the simulations the transmission of a 4-PAM (pulse amplitude modulation) signal with statistics $r = 8.2$, $\sigma_\beta^2 = 28.8$, and $\bar{\gamma} = 6.8$ over the channel $\mathbf{h} = [0.25\ 0.64\ 0.80\ -0.55]^T$ in the absence of noise, and an equalizer with $M = 20$ coefficients implemented as a $T/2$-FSE, initialized with only one nonzero and unitary element in the tenth position.

Fig. 2-(a) shows the theoretical EMSE along the iterations using (15) and its experimental value estimated from the ensemble-average of 500 independent runs of a DM-CMA equalizer in a stationary environment with $\mu = 0.05$. To facilitate the visualization, the experimental curves were filtered by a moving-average filter with 16 coefficients. The steady-state EMSE values predicted by (16) and (19) are also shown in the figure. In this situation, both steady-state analyses inside the ROI present a similar result and there is a good agreement between the transient analysis and the simulation. Considering a larger step-size ($\mu = 0.2$), the transient analysis is not as accurate as in the previous situation, as shown in Fig. 2-(b). Since (16) was derived from the transient analysis, it is also not accurate to predict the steady-state EMSE. On the other hand, the analytical expression for the steady-state EMSE obtained via the energy conservation method (Eq. (19)) provides a good agreement with the experimental result.

To verify the accuracy of the DM-CMA steady-state analyses, we assume a non-stationary environment with $\mathbf{Q} = 10^{-6}\mathbf{R}$. Fig. 3-(a) shows the theoretical curves of the EMSE predicted by (16) and (19) inside the ROI and (21) outside the ROI, and also the experimental values estimated through an ensemble-average of 50 independent runs. Fig. 3-(b) shows the probability of divergence (P_d) of NCMA, which is obtained from $L = 50$ repetitions of each experiment, starting from the same initial condition $\mathbf{w}(0)$. As in [13], we label a given run of NCMA as "diverging" if $y(n)$ overflows (we check for NaNs in Matlab). Then, we compute the probability of divergence as $P_d = (\text{Number of curves diverging})/L$. In the interval of μ where the probability of divergence is almost null, DM-CMA works inside the ROI and for lower step-sizes, the theoretical results from (16) and (19) are similar. However, as the step-size becomes larger, the theoretical result from (19) is more accurate than that of (16). It is relevant to notice that (16) allows to determine a more accurate interval of μ for which the probability of divergence of NCMA is almost null. Note that the maximum of the curve, which occurs when the denominator of (16) is null, coincides with the transition of the curve of P_d shown in Fig. 3-(b). The values of EMSE predicted by (16) and (19) are negative for $\mu > 0.39$ and $\mu > 0.49$, respectively, and therefore, are not shown in the figure. In the interval of μ where the probability of divergence is higher, DM-CMA works outside the ROI during a larger number of iterations. In this case, the experimental EMSE is bounded by the theoretical curve predicted by (21), assuming the operation outside the ROI.

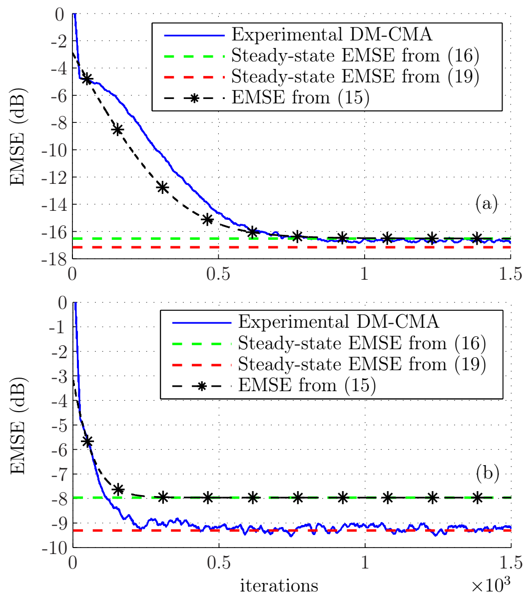

Fig. 2. Theoretical and experimental EMSE along the iterations for DM-CMA, assuming a) $\mu = 0.05$ and b) $\mu = 0.2$; $\delta = 10^{-5}$ $\mathbf{Q} = 0$, 4-PAM, $M = 20$; ensemble-average of 500 independent runs.

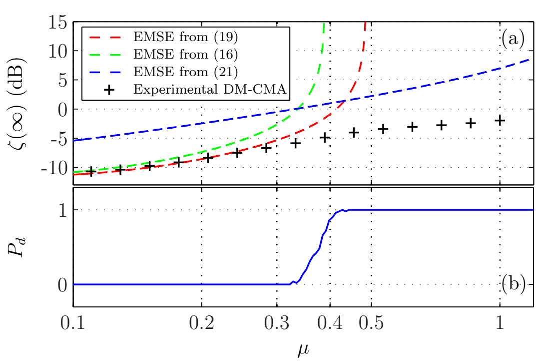

Fig. 3. (a) Theoretical and experimental steady-state EMSE for DM-CMA; (b) Probability of divergence of NCMA as a function of μ; 4-PAM, $M = 20$, $\delta = 10^{-5}$; $\mathbf{Q} = 10^{-6}\mathbf{R}$; ensemble average of 50 independent runs.

VI. Conclusion

In this paper, we presented a statistical analysis for DM-CMA. Assuming that the algorithm operates inside the ROI, we provided a transient analysis, which showed a good agreement with simulations, mainly for small step-sizes. The steady-state EMSE inside the ROI was then obtained as the limiting case of the transient analysis. The proposed model provides a reasonable estimate for the range of step-sizes in which the probability of divergence of NCMA is approximately zero. Using the energy conservation method, we obtained another expression for the steady-state EMSE, which is more accurate for larger step-sizes. In order to obtain an expression for the tracking EMSE in a worst case scenario, we assumed that DM-CMA operated only outside the ROI. The resulting expression in conjunction with those obtained when DM-CMA operates inside the ROI give a range of values for the steady-state EMSE of DM-CMA in all possible situations.

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