# Channel equalization for synchronization of Ikeda maps

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## 1. Introduction

This paper proposes **an adaptive equalization scheme to recover a binary sequence modulated by a chaotic signal**, which in turn is generated by **Ikeda maps**. The proposed scheme employs the **normalized least-mean-squares (NLMS) algorithm with a modification** to enable **chaotic synchronization even when the communication channel is not ideal**.

This chaotic communication system

1. provides an **alternative spread spectrum modulation** that improves the **level of privacy** in data transmission;
2. may present the **properties of spread spectrum modulations**, as multipath and jamming immunity.

## 2. Problem Formulation

- The binary signal $m(n) \in \{-1, +1\}$ is encoded by using the second component of the state vector $\mathbf{x}(n)$, i.e.,

$$s(n) = m(n)x_2(n)$$

- The signal $s(n)$ is fed back and transmitted through a communication channel (ISI + noise)
- The adaptive equalizer is an $M$-tap FIR filter with input regressor vector $\mathbf{r}(n)$ and output

$$\widehat{s}(n) = \mathbf{r}^T(n)\mathbf{w}(n-1)$$

- $\Delta$ is the delay of the cascade channel + equalizer
- If transmitter and receiver completely synchronize, $\widehat{\mathbf{x}}(n) \to \mathbf{x}(n)$ and the information signal can be decoded via

$$\widehat{m}(n) \triangleq \widehat{s}(n)/\widehat{x}_2(n)$$

where $\widehat{x}_2(n)$ is the estimate of $x_2(n)$ and the second component of the state vector $\widehat{\mathbf{x}}(n)$
- The equalizer is adapted in a supervised manner through the error

$$e(n) = m(n-\Delta) - \widehat{m}(n)$$

where $\{m(n-\Delta)\}$ is a training sequence
- The equations governing the global dynamical system have the following form (Ikeda map)

$$\text{Master:}\quad \mathbf{x}(n) = \mathbf{A}_t(n)\mathbf{x}(n-1) + [\,R\;\;0\,]^T$$
$$\text{Slave:}\quad \widehat{\mathbf{x}}(n) = \mathbf{A}_r(n)\widehat{\mathbf{x}}(n-1) + [\,R\;\;0\,]^T$$

where $\mathbf{x}(n) \triangleq [x_1(n)\; x_2(n)]^T$, $\widehat{\mathbf{x}}(n) \triangleq [\widehat{x}_1(n)\; \widehat{x}_2(n)]^T$, and $R$ is a constant.

In the chaotic signal generator (CSG) of the transmitter, we have

$$\mathbf{A}_t(n) = C_2\begin{bmatrix}\cos\theta_n & -\sin\theta_n\\ \sin\theta_n & \cos\theta_n\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & m(n-1)\end{bmatrix},$$

where

$$\theta_n = C_1 - \frac{C_3}{1+x_1^2(n-1)+x_2^2(n-1)m^2(n-1)},$$

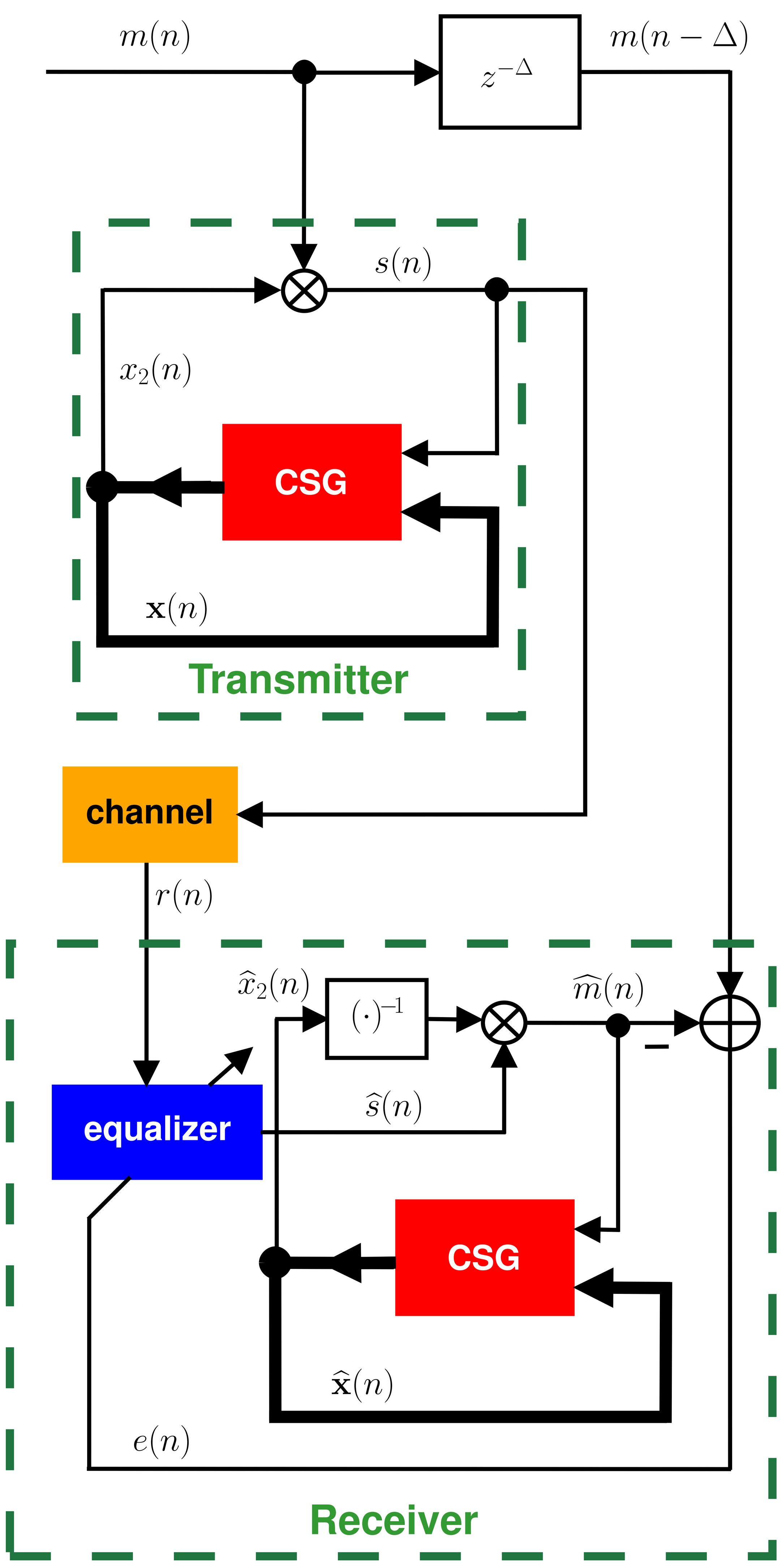
$C_i,\ i=1,2,3$ are constant parameters of the Ikeda map and $m^2(n-1)=1$.

In the CSG of the receiver, we have

$$\mathbf{A}_r(n) = C_2\begin{bmatrix}\cos\widehat{\theta}_n & -\sin\widehat{\theta}_n\\ \sin\widehat{\theta}_n & \cos\widehat{\theta}_n\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \widehat{m}(n-1)\end{bmatrix},$$

where

$$\widehat{\theta}_n = C_1 - \frac{C_3}{1+\widehat{x}_1^2(n-1)+\widehat{x}_2^2(n-1)\widehat{m}^2(n-1)}.$$




Chaotic communication system

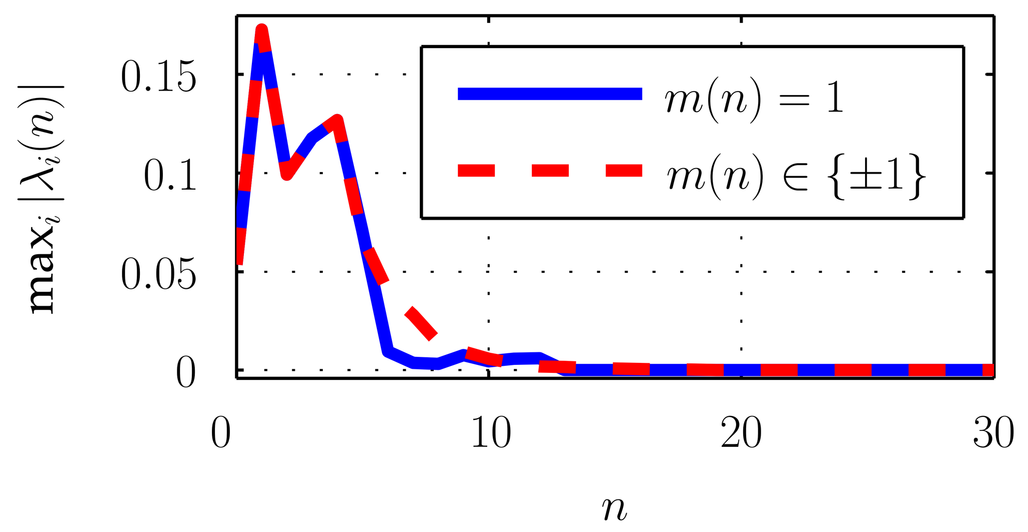
## 3. Synchronization for an ideal channel

The synchronization error is defined as $\boldsymbol{\xi}(n) \triangleq \widehat{\mathbf{x}}(n) - \mathbf{x}(n)$, which can be rewritten as

$$\boldsymbol{\xi}(n) = [\mathbf{A}_r(n) - \mathbf{A}_t(n)]\,\boldsymbol{\xi}(n-1)$$

Master and slave are said **completely synchronized** if $\boldsymbol{\xi}(n)\to\mathbf{0}$ as $n$ grows. Consequently, they synchronize completely if the eigenvalues of $[\mathbf{A}_r(n) - \mathbf{A}_t(n)]$ satisfy $|\lambda_i(n)| < 1,\ i=1,2,$ for all $n$.

In the figure below, we show a numerical simulation to illustrate that the **synchronization between master and slave may be achieved for an ideal channel**, considering two situations: $m(n)=1$ and a binary equiprobable random message, $m(n) \in \{-1, +1\}$.




Maximum absolute value of the eigenvalues of $[\mathbf{A}_r(n) - \mathbf{A}_t(n)]$ along the iterations; Ikeda map ($\mathbf{x}(0)=\mathbf{0}$; $\widehat{\mathbf{x}}(0)=[0.1\;\;-0.1]^T$; $C_1=0.4$, $C_2=0.9$, $C_3=6$, and $R=1$)

**We have synchronization for both cases**, since $\max_i|\lambda_i(n)| < 1,\ \forall n$ for the two cases considered.

## 4. The cNLMS algorithm

To obtain the stochastic gradient algorithm shown in table below, we define the following instantaneous cost-function

$$\hat{J}(n) = e^2(n) = [m(n-\Delta) - \widehat{m}(n)]^2$$

and follow the same steps for obtaining the NLMS algorithm.

Initialize the algorithm by setting:

$\mathbf{w}(-1)=\mathbf{0}$, $\widehat{\mathbf{x}}(0)=[0.1\;\;-0.1]^T$, $\mathbf{b}=[\,R\;\;0\,]^T$

$\delta, \varepsilon$: small positive constants

$X$: large positive constant

For $n = 0, 1, 2, 3\ldots$, compute:

$\widehat{s}(n) = \mathbf{r}^T(n)\mathbf{w}(n-1)$

To avoid wrong estimates when $\widehat{x}_2(n)$ is too large

if $|\widehat{x}_2(n)| > X$

$\widehat{x}_2(n) \leftarrow X\,\text{sign}[\widehat{x}_2(n)]$

end

To prevent division by a value close to zero

if $|\widehat{x}_2(n)| \le \varepsilon$

$\widehat{m}(n) = \text{sign}[\widehat{s}(n)\,\widehat{x}_2(n)]$

else

$\widehat{m}(n) = \dfrac{\widehat{s}(n)}{\widehat{x}_2(n)}$

end

$e(n) = m(n-\Delta) - \widehat{m}(n)$

$\mathbf{w}(n) = \mathbf{w}(n-1) + \dfrac{\tilde{\mu}_c}{\delta + \|\mathbf{r}(n)\|^2}\widehat{x}_2(n)e(n)\mathbf{r}(n)$

$\widehat{\theta}_{n+1} = C_1 - \dfrac{C_3}{1+\widehat{x}_1^2(n)+\widehat{x}_2^2(n)\widehat{m}^2(n)}$

$\mathbf{A}_r(n+1) = C_2\begin{bmatrix}\cos\widehat{\theta}_{n+1} & -\sin\widehat{\theta}_{n+1}\\ \sin\widehat{\theta}_{n+1} & \cos\widehat{\theta}_{n+1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \widehat{m}(n)\end{bmatrix}$

$\widehat{\mathbf{x}}(n+1) = \mathbf{A}_r(n+1)\,\widehat{\mathbf{x}}(n) + \mathbf{b}$

end

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# 5. Stability conditions

The update equation of cNLMS can be rewritten as

$$\mathbf{w}(n) = \left[\mathbf{I} - \frac{\tilde{\mu}}{\delta + \|\mathbf{r}(n)\|^2}\mathbf{r}(n)\mathbf{r}^T(n)\right]\mathbf{w}(n-1) + \tilde{\mu}\,\hat{x}_2(n)m(n)\frac{\mathbf{r}(n)}{\delta + \|\mathbf{r}(n)\|^2} \qquad (\star)$$

The matrix between brackets has $M-1$ eigenvalues equal to one and one eigenvalue equal to

$$\lambda_1 = 1 - \tilde{\mu}\,\mathbf{r}^T(n)\mathbf{r}(n)/[\delta + \|\mathbf{r}(n)\|^2].$$

Noticing that

$$0 \le \frac{\mathbf{r}^T(n)\mathbf{r}(n)}{\delta + \|\mathbf{r}(n)\|^2} < 1,$$

and for $\|\mathbf{r}(n)\|^2 \gg \delta$, $\mathbf{r}^T(n)\mathbf{r}(n)/(\delta + \|\mathbf{r}(n)\|^2) \approx 1$, in order to ensure $|\lambda_1| < 1$, we must choose $\tilde{\mu}$ in the interval

$$\boxed{0 < \tilde{\mu} < 2}$$

The norm of the second term of the r.h.s. of $(\star)$ is bounded since

$$0 \le \tilde{\mu}\,|\hat{x}_2(n)|\,|m(n)|\frac{\|\mathbf{r}(n)\|}{\delta + \|\mathbf{r}(n)\|^2} \le \tilde{\mu}\,X\frac{\sqrt{\delta}}{2\delta} < \infty.$$

Therefore, using (deterministic) exponential stability results for the LMS algorithm, we conclude that **cNLMS is stable in a robust sense if $\tilde{\mu}$ is chosen in the interval** $0 < \tilde{\mu} < 2$.

# 6. Simulation results

- The parameters of the Ikeda map were set as $C_1 = 0.4$, $C_2 = 0.9$, $C_3 = 6$, and $R = 1$
- The state vectors were initialized as $\mathbf{x}(0) = \mathbf{0}$ and $\hat{\mathbf{x}}(0) = [0.1 \quad -0.1]^T$
- We assume the transmission of a binary sequence $m(n) \in \{-1, 1\}$
- The equalizers were initialized as $\mathbf{w}(0) = \mathbf{0}$
- For comparison, we also consider the chaotic communication system without equalizer, in which $\hat{s}(n) = r(n)$

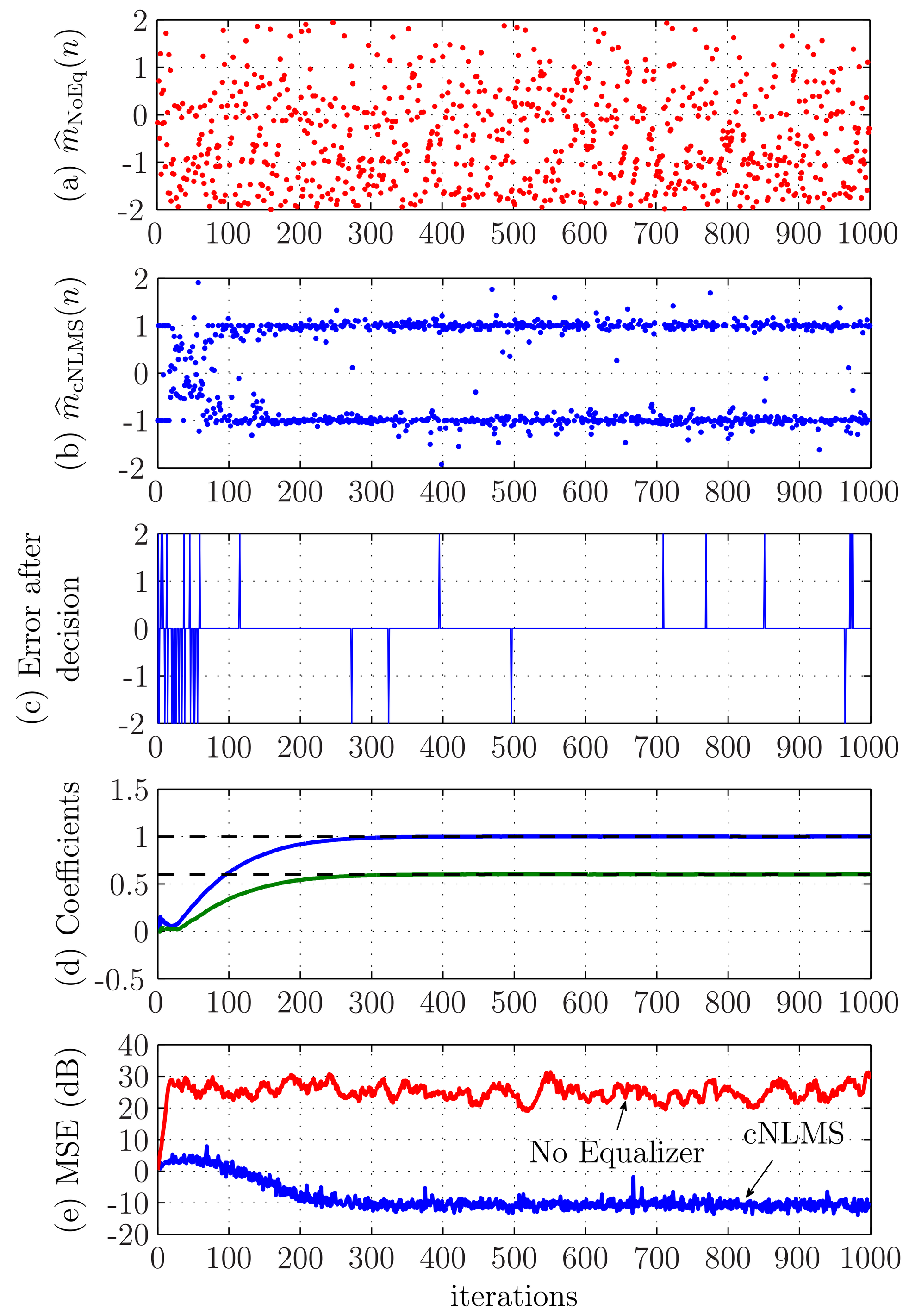
## Scenario 1

The encoded sequence $s(n)$ is transmitted through the channel

$$H_1(z) = \frac{1}{1 + 0.6z^{-1}}$$

with $\mathrm{SNR} = 30$ dB, $\Delta = 0$ and $M = 2$.

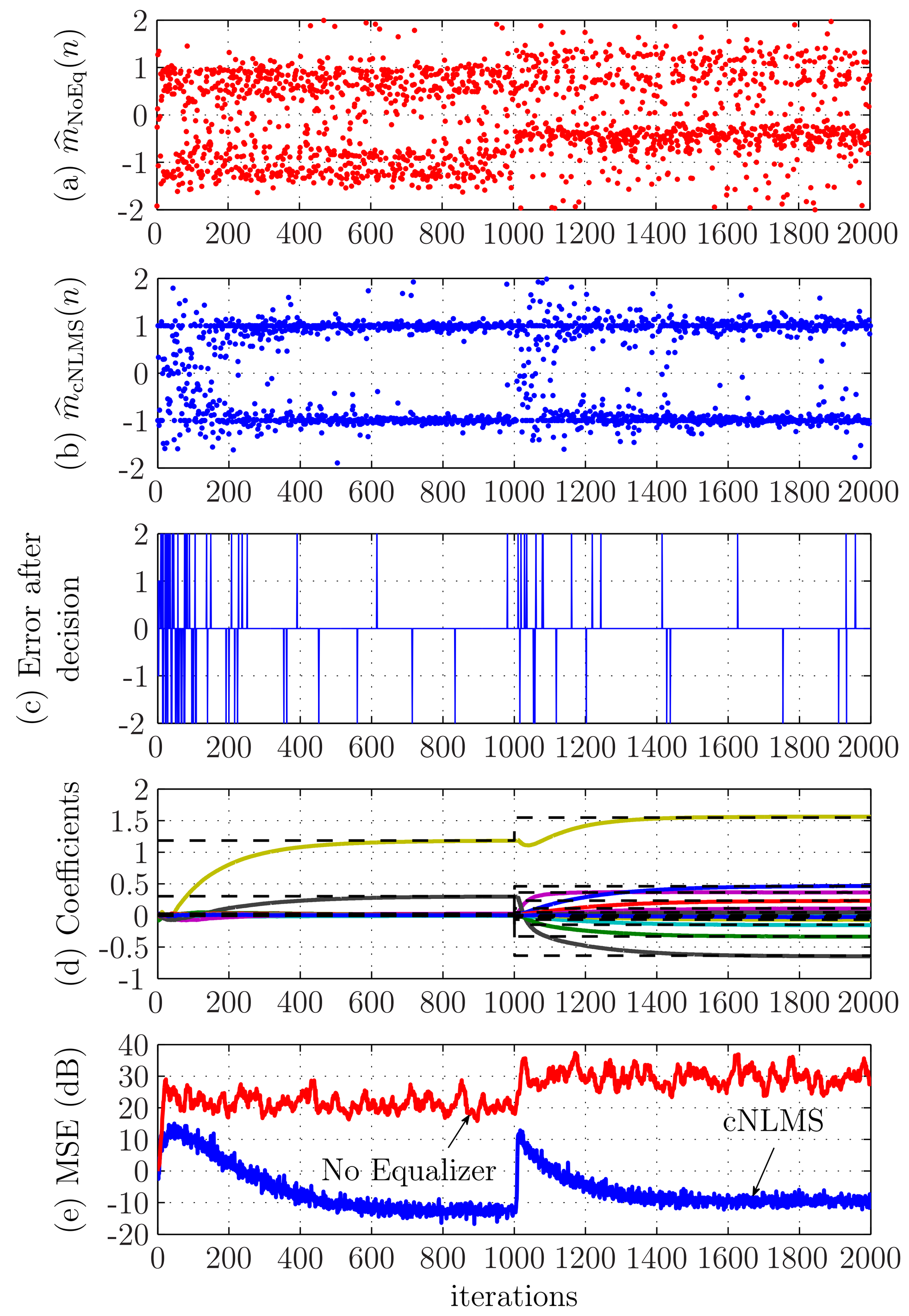
**cNLMS approaches to** $\mathrm{w}_o \approx [1 \quad 0.6]^T$ and therefore, **the equalizer is working as expected** since this solution mitigates the intersymbol interference, recovering properly the transmitted sequence. The communication is completely lost in the case with no equalizer.




Estimated sequence with (a) No equalizer and (b) cNLMS ($\mu$=0.1, $\delta$=$10^{-2}$, $\varepsilon$=0.1); (c) Errors after decision; (d) Average of the coefficients of cNLMS and Wiener (dashed lines); (e) Estimated cMSE; average of 1000 runs.

## Scenario 2

The encoded sequence $s(n)$ is transmitted initially through the real part of the telephonic channel [Picchi & Prati, 1987] and changed to its imaginary part at $n = 1000$, with $\mathrm{SNR} = 30$ dB, $\Delta = 8$, and $M = 15$.

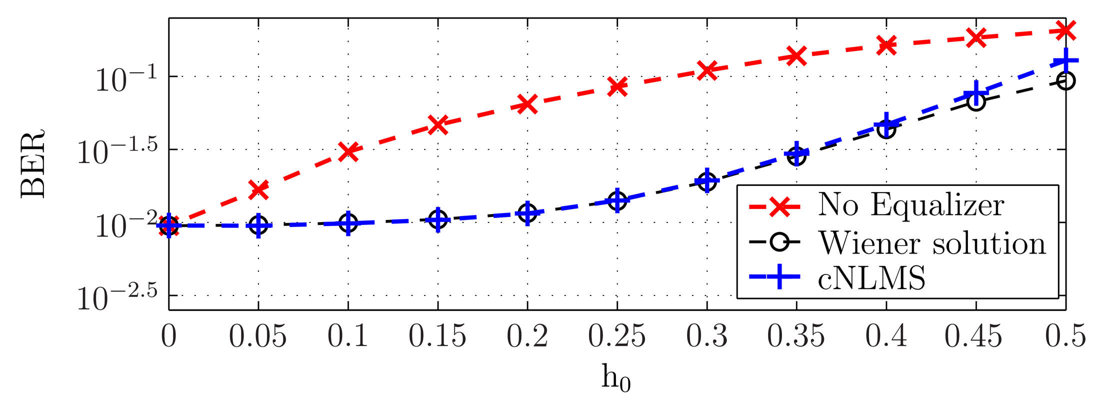



Estimated sequence with (a) No equalizer and (b) cNLMS ($\mu$=0.5, $\delta$=$10^{-5}$, $\varepsilon$=0.1); (c) Errors after decision; (d) Average of the coefficients of cNLMS and Wiener (dashed lines); (e) Estimated cMSE; average of 1000 runs.

**cNLMS converges to the Wiener solution and is able to track the abrupt variation in the channel**, leading approximately 600 iterations to achieve the steady-state again. The equalizer plays an important role to mitigate the intersymbol interference since the performance of the system without equalizer is much worse.

## Scenario 3

We assume $\mathrm{SNR} = 30$ dB, $M = 5$, $\Delta = 3$, and the channel $H_3(z) = h_0 + z^{-1} + h_0 z^{-2}$, $0 \le h_0 \le 0.5$ to obtain BER curves as a function of $h_0$. The smaller the value of $h_0$ the lower the intersymbol interference introduced by the channel.




Bit error rate as a function of the channel $H_3(z) = h_0 + z^{-1} + h_0 z^{-2}$ with $\mathrm{SNR} = 30$ dB; cNLMS ($\mu$=0.1, $\delta$=$\varepsilon$=$10^{-5}$).

**cNLMS outperforms the case with no equalizer for $h_0 > 0$.** The relatively high $\mathrm{BER} \approx 10^{-2}$ in the left of the figure is only due to channel noise and reflects the extreme sensitivity of chaotic synchronization to noise. **The issue of channel equalization was successfully solved as shows the almost coincidence of the cNLMS and Wiener solution curves.**

# 7. Conclusions

In this paper, we proposed a supervised equalization scheme based on the NLMS algorithm for recovering a binary sequence in chaos-based digital communication systems. The main conclusions are:

- simulations show that **the proposed algorithm can successfully permit chaotic communications**;
- this is the first **adaptive scheme proposed for the chaotic modulation in which the message is fed back into the CSG**; and
- although we considered the Ikeda map in the simulations, **the cNLMS algorithm can be also used with other chaotic maps** (e.g., Hénon map).

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