On Combinations of CMA Equalizers

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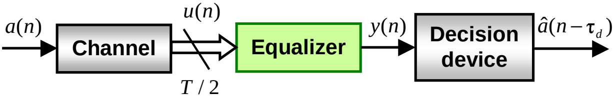
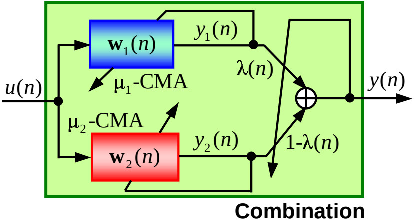
Problem Formulation





$$y(n) = \lambda(n)y_1(n) + [1 - \lambda(n)]y_2(n)$$

$$\mathbf{w}_i(n+1) = \mathbf{w}_i(n) + \mu_i e_i(n)\mathbf{u}(n),\ i = 1,2$$

$$e_i(n) = [r - y_i^2(n)]y_i(n),\quad i = 1,2$$

$$r \triangleq \mathrm{E}\{a^4(n)\}/\mathrm{E}\{a^2(n)\}$$

Adaptations of the mixing parameter

Convex Combination [Arenas-García, Figueiras-Vidal, 2006]

$$\lambda(n) = \left\{1 + \mathrm{e}^{-\alpha(n)}\right\}^{-1}$$

$$\alpha(n+1) = \alpha(n) + \mu_\alpha e_\alpha(n)\lambda(n)[1-\lambda(n)]$$

$$e_\alpha(n) = \left[r - y^2(n)\right] y(n)[y_1(n) - y_2(n)]$$

In the affine combination $\lambda(n)$ is not restricted to $[0,1]$ [Bershad, Bermudez, Tourneret, 2008].

Affine Combination

$$\lambda(n+1) = \lambda(n) + \mu_\lambda e_d(n)[y_1(n) - y_2(n)]$$

$$e_d(n) = \hat{a}(n - \tau_d) - y(n)$$

Tracking analysis

Random-walk model

$$\mathbf{w}_o(n+1) = \mathbf{w}_o(n) + \mathbf{q}(n)$$
$$\mathbf{Q} = \mathrm{E}\{\mathbf{q}(n)\mathbf{q}^T(n)\}$$

EMSE (ζ_{11} or ζ_{22}) and cross-EMSE (ζ_{12})

$$\zeta_{ij} \approx \frac{\mu_i \mu_j \sigma_\beta^2 \mathrm{Tr}(\mathbf{R}) + \mathrm{Tr}(\mathbf{Q})}{\bar{\gamma}(\mu_i + \mu_j) - \mu_i \mu_j \mathrm{Tr}(\mathbf{R})\xi}, \quad i,j = 1,2$$

$\mathrm{Tr}(\cdot)$: trace of a matrix

$$\mathbf{R} = \mathrm{E}\{\mathbf{u}(n)\mathbf{u}^T(n)\}$$
$$\sigma_\beta^2 = \mathrm{E}\{a^6(n) - r^2 a^2(n)\}$$
$$\bar{\gamma} = 3\mathrm{E}\{a^2(n)\} - r$$
$$\xi = r(3\mathrm{E}\{a^2(n)\} + r)$$

Analytical models for the combination at the steady-state

From the derivative of CM cost function, we obtain

Optimum mixing parameter: $\bar{\lambda}_o(\infty) \triangleq \lim_{n\to\infty} \mathrm{E}\{\lambda(n)\} \approx \dfrac{\Delta\zeta_2}{\Delta\zeta_1 + \Delta\zeta_2}$

Steady-state EMSE of the combination: $\zeta \approx \zeta_{12} + \dfrac{\Delta\zeta_1 \Delta\zeta_2}{\Delta\zeta_1 + \Delta\zeta_2}$

where $\Delta\zeta_i \triangleq \zeta_{ii} - \zeta_{12}, \; i = 1{,}2.$

Remarks

- These expressions were first obtained in [Arenas-García, Figueiras-Vidal, Sayed, 2006] for the convex combination of two LMS filters.
- In the affine combination, $\lambda(n)$ and consequently $\bar{\lambda}_o(\infty)$ are not restricted to the interval $[0, 1]$.

Theoretical results for the affine combination in a stationary environment

Defining $\delta \triangleq \mu_2/\mu_1$, with $0 < \delta < 1$, we obtain:

Optimum mixing parameter

$$\bar{\lambda}_{\mathrm{o}}(\infty) \approx \frac{\delta \left[2 - \mu_1 \mathrm{Tr}(\mathbf{R})\xi\, \bar{\gamma}^{-1}\right]}{2\,(\delta - 1)}$$

To ensure the stability of μ_1-CMA, $0 < \mu_1 < 2\bar{\gamma}/(3\mathrm{Tr}(\mathbf{R})\xi)$ must be satisfied. Hence, $\bar{\lambda}_{\mathrm{o}}(\infty)$ is always negative.

EMSE of the combination

$$\zeta \approx \frac{1}{2}\,\frac{\mu_2 \sigma_\beta^2 \mathrm{Tr}(\mathbf{R})}{(1+\delta)\bar{\gamma} - \mu_2 \mathrm{Tr}(\mathbf{R})\xi}$$

For $\delta \to 1$, the affine combination provides a 3dB gain. In this case, $\bar{\lambda}_{\mathrm{o}}(\infty) \to -\infty$.

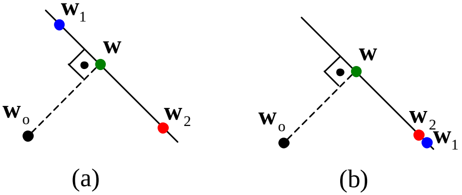
Understanding the mixing parameter

- The overall steady-state error can be rewritten as

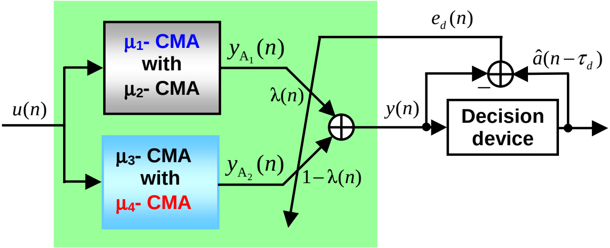
$$e(n) = \underbrace{e_2(n)}_{d(n)} + \lambda(n) \underbrace{\gamma(n)[\mathbf{w}_2(n) - \mathbf{w}_1(n)]^T \mathbf{u}(n)}_{-x(n)}$$

  where $\gamma(n) = 3a^2(n - \tau_d) - r$, $d(n)$ is the signal to be estimated and $x(n)$ plays the role of input signal.

- Assuming that $\mathbf{w}_i$, $i = 1{,}2$ vary slowly compared to λ, this equation has a simple geometric interpretation:



Improving the EMSE reduction in a stationary environment



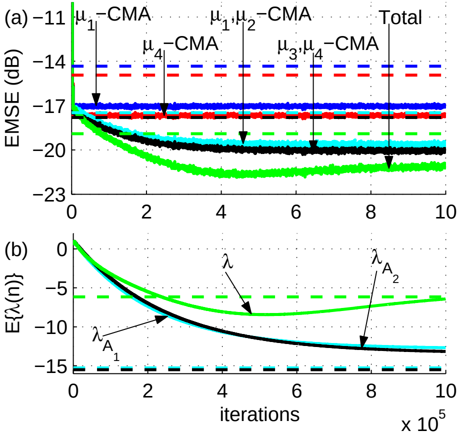
$$\mu_2 = \delta_1 \mu_1, \quad 0 \ll \delta_1 < 1$$
$$\mu_4 = \delta_2 \mu_3, \quad 0 \ll \delta_2 < 1$$

$$\lim_{(\delta_1,\delta_2)\to(1,1)} \zeta \approx \frac{3}{8}\zeta_{11}.$$

This represents an EMSE reduction of 4.26 dB

Simulation Results - stationary case

$\mu_1 \approx \mu_2 \approx \mu_3 \approx \mu_4$



(a) Theoretical and experimental EMSE (b) Ensemble-average of $\lambda(n)$, $\lambda_{A_1}(n)$, $\lambda_{A_2}(n)$ and theoretical value of $\bar{\lambda}_o(\infty)$

Theoretical results for the affine combination in nonstationary environments

The largest EMSE reduction occurs when $\zeta_{11} \approx \zeta_{22}$. This can happen when

1) $\mathrm{Tr}(\mathbf{Q}) \approx \mu_1 \mu_2 \sigma_\beta^2 \mathrm{Tr}(\mathbf{R})$

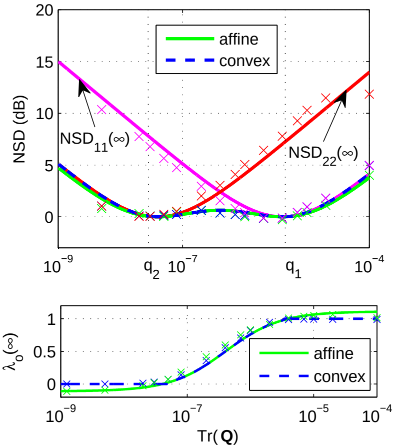
$$\frac{\zeta}{\zeta_{22}} \approx \frac{1}{2} + \frac{2\delta}{(\delta+1)^2},$$

or

2) $\mu_1 \approx \mu_2$

$$\lim_{\delta \to 1} \zeta = \frac{\zeta_{22}}{2} + \frac{\sigma_\beta^2 \mathrm{Tr}(\mathbf{R}) \mathrm{Tr}(\mathbf{Q})}{2\bar{\gamma}^2 \zeta_{22}}$$

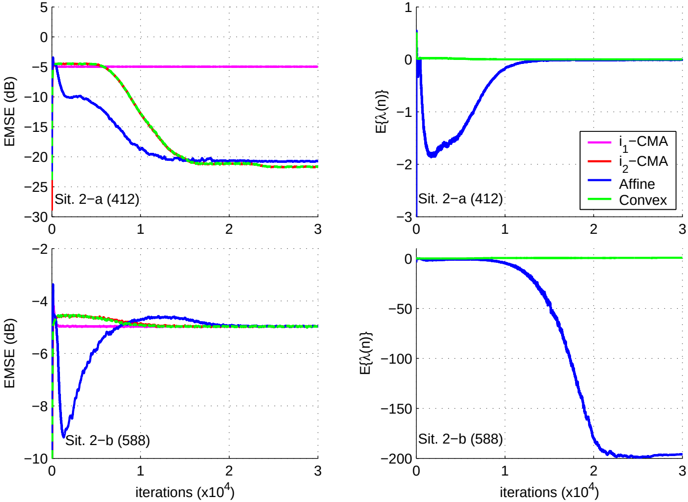
In both cases, the EMSE reduction is limited by 3 dB



- $\mathrm{NSD}_{ii}(\infty) = \zeta_{ii}/\zeta_{\mathrm{o}},\ \ i = 1{,}2,\ \mathrm{NSD}(\infty) = \zeta/\zeta_{\mathrm{o}}$, where ζ_{o} is the optimum steady-state EMSE of a CMA equalizer.

Combination of 2 CMAs with different initializations

- BPSK, Channel $H(z) = [1 + 0{,}6z^{-1}]^{-1}$; SNR= 25 dB
- $\mathbf{w}_1(0) = [\ 0{,}40 \quad 0{,}05\ ]^T$ and $\mathbf{w}_2(0) = [\ 0{,}40 \quad -0{,}60\ ]^T$



Conclusions

Through the analysis and simulations, we observe that

- When the component equalizers have close step-sizes, the affine combination can provide an EMSE reduction of 3 dB.
- In a nonstationary environment, the minimum steady-state EMSE of the combination is equal to the steady-state EMSE of a CMA equalizer with optimal step-size. Thus, the affine combination can perform similarly to the convex combination.
- To avoid local minima, we combined two CMAs with different initializations. There may exist situations where the combined scheme avoids local minima. Comparing to the convex combination, the affine combination may present faster convergence and search a minimum more efficiently.